Lecture 3

Brief Review of Topology and Geometry I
-- Topological Numbers & Differential Forms

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Prologue：
Paradigms in Condensed Matter Physics

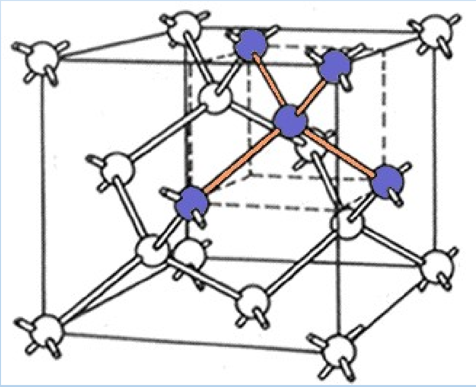
The Search for New States/Phases of Matter

The search for new elements led to a golden age of chemistry.

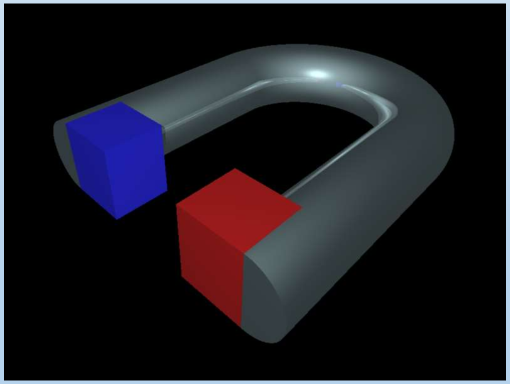
The search for new particles led to the golden age of particle physics.

Now in a golden age of condensed matter physics, we ask:
what are the possible fundamental states of matter?

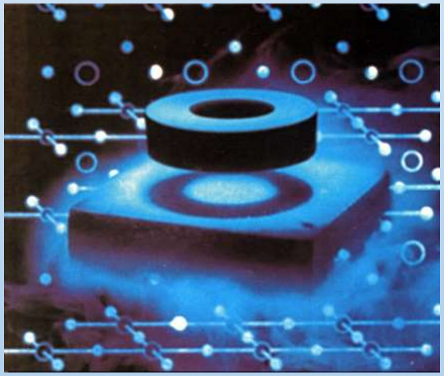
(Known in early 20th century)



Crystal: Broken translational symmetry



Magnet: Broken rotational symmetry



Superconductor: Broken gauge symmetry

Landau-Ginzburg-Wilson Paradigm (CMT)

- **Classical Phase Transitions:**
  - ✓ **Associated with symmetry breaking**
  - ✓ **Characterized by local order parameter(s)**
  - ✓ **Disorder-Order Transition driven by thermal fluctuations**

  Examples: superfluids, ferromagnetism, superconductivity

- **Landau-Fermi Liquid Theory:**
  - ✓ **Quasi-particles are fermions (existence of Fermi surface)**
  - ✓ **Quasiparticles have same charge and spin (quantum numbers) as electrons**
  - ✓ **Electron interactions incorporated in energy as functional of quasiparticle occupation number (quasiparticle energy and Landau parameters)**

  Examples: Helium 3, many metals etc

- **Common Feature: Can be understood by Renormalization Group Flow Fixed Points (Wilson & Shankar)**

Topological Order: Beyond the Landau Paradigm

- **Novel phases at T=0 due to quantum effects (quantum matter)**
- **No symmetry breaking, no local order parameter(s)**
- **Characterized by a topological number**
- **Robust against weak disorders and interactions**
- **Correspondence between bulk and edge (in 2d) /surface (in 3d)**
- **Topology-dependent ground state degeneracy**
- **Fractionalization of quantum numbers (of quasiparticles)**
- **Fractional (exchange and exclusion) statistics of quasiparticles**
- **Intricate interplay between symmetries and topological orders**

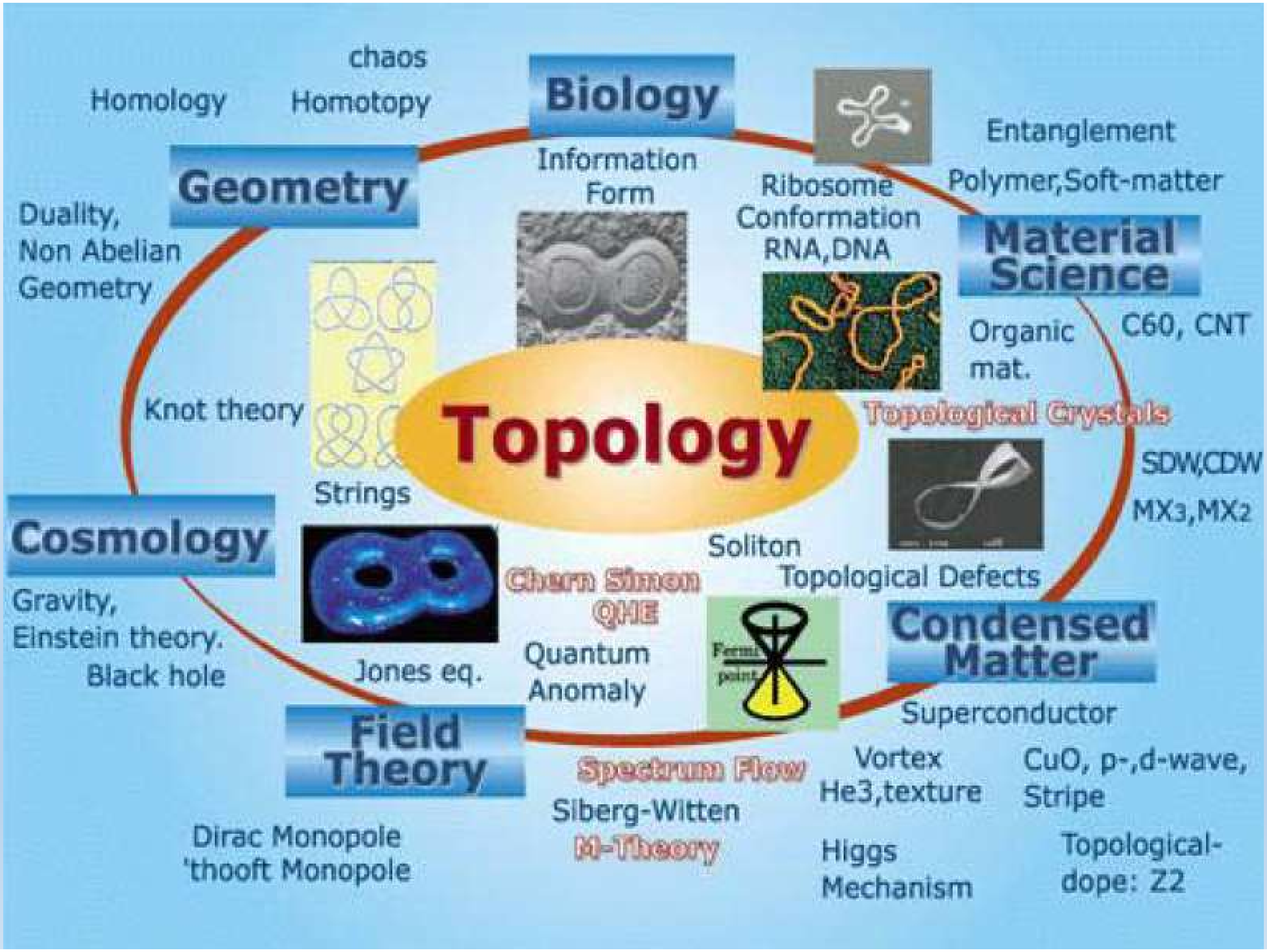
- **Examples: quantum Hall effect, Mott insulators, quantum spin Hall effect, quantum spin liquids,topological insulators/superconductors, Dirac/Weyl semi-metals, connection w/ fundamental Physics etc**

Respondence to Professor Wen's Classes

- Class 1: Symmetry Breaking in T=0 Quantum Phase Transition
  (with the example of the 1d Transverse Ising Model)

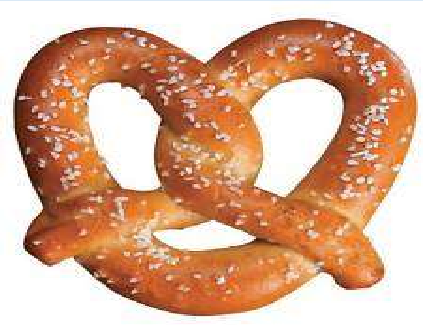
- Class 2: Topological Order (beyond Landau's paradigm)
  (in the context of String-Net Models)

A Primer of Topology

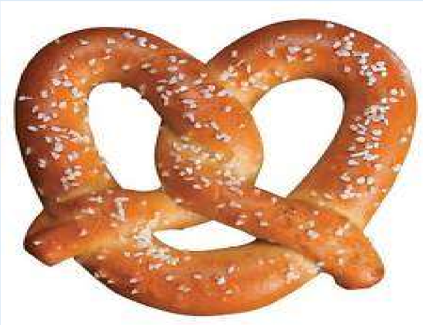


chaos
Homology
Homotopy
Biology
Entanglement
Polymer,Soft-matter
Geometry
Information
Form
Ribosome
Conformation
RNA,DNA
Duality,
Non Abelian
Geometry
Material
Science
Organic
mat.
C60, CNT
Knot theory
Topology
Topological Crystals
SDW,CDW
MX3,MX2
Strings
Cosmology
Soliton
Topological Defects
Chern Simon
QHE
Gravity,
Einstein theory.
Black hole
Condensed
Matter
Jones eq.
Quantum
Anomaly
Fermi
point
Superconductor
Field
Theory
Spectrum Flow
Vortex
He3,texture
CuO, p-,d-wave,
Stripe
Siberg-Witten
M-Theory
Dirac Monopole
'thooft Monopole
Higgs
Mechanism
Topological-
dope: Z2

Surface of Orange, Mug and Pretzel

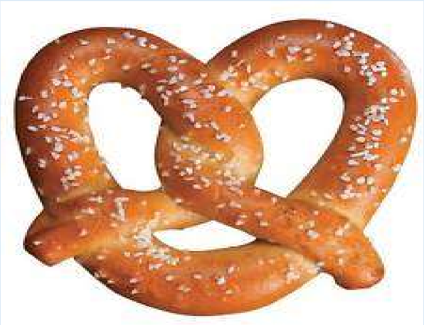


Surface of Orange, Mug and Pretzel



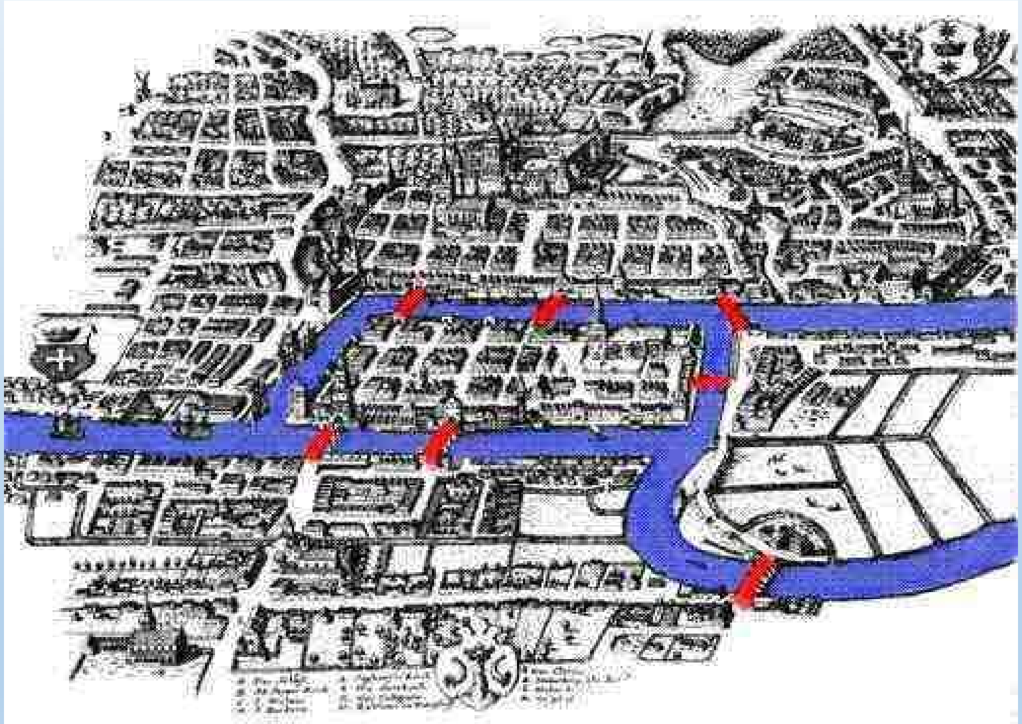
Topology: Properties unchanged under continuous deformations

Surface of Orange, Mug and Pretzel



How to describe and characterize topological differences between these three things?

Königsberg Seven Bridge Problem



[source: MacTutor History of Mathematics Archive]

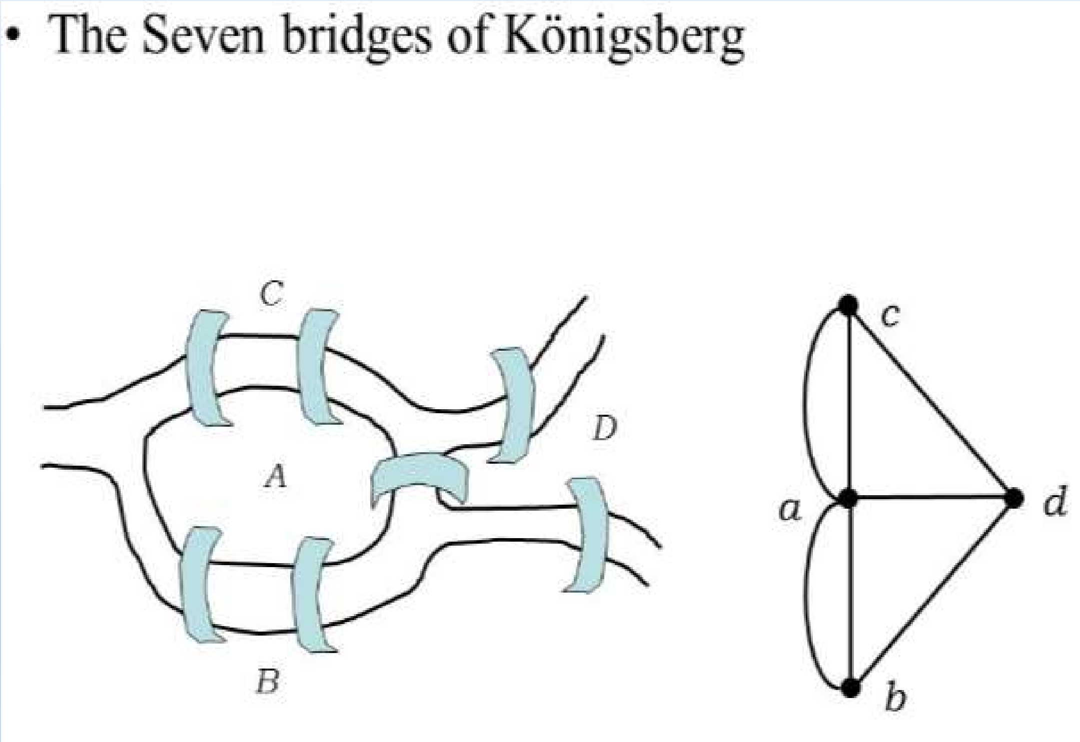
Problem:

Is it possible for a walker to go through all several bridges only once and to return to the starting position?

Euler's Solution of the Seven Bridge Problem

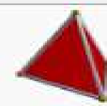
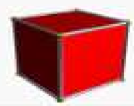
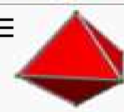
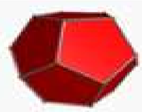
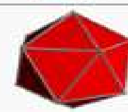
Swistzland Stamp (2207)

Impossible! Because each vertex is connected to an odd number of links.



Credit: Aiden Ball

Euler Characteristics for Polyhedra

Name	Image	Vertices V	Edges E	Faces F	Euler characteristic: $V - E + F$
Tetrahedron		4	6	4	2
Hexahedron or cube		8	12	6	2
Octahedron		6	12	8	2
Dodecahedron		20	30	12	2
Icosahedron		12	30	20	2

(Credit: Wikipedia)

$\chi(M) \equiv V-E+F$ (M: 2d closed surface)

Actually true for any Convex/Spherical polyhedra!

Gauss-Bonnet Theorem

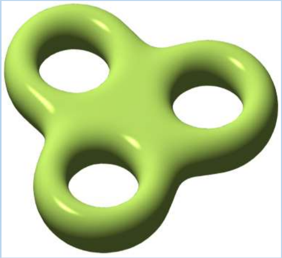
$$\frac{1}{2\pi}\int_M K dA = \chi(M) = 2(1-g).$$

where M: 2d closed surface

K: Gauss Curvature , g: genus (# handles)

$g = 0$

$g = 1$



$g = 3$

Ideas inspired by this Line of Thoughts

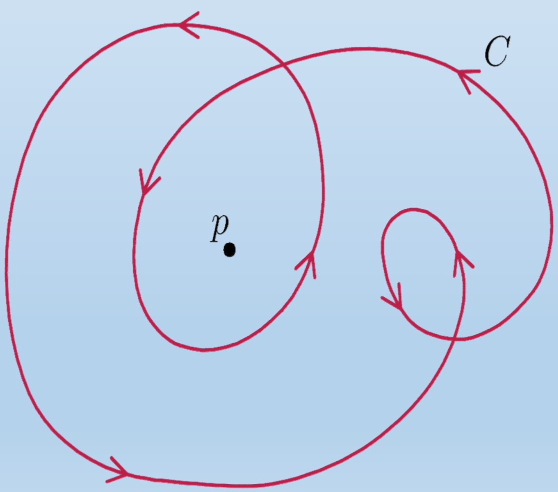
- Discrete approach can be Exact for Topology of a Continuous Object
  (Combinatoric Topology)

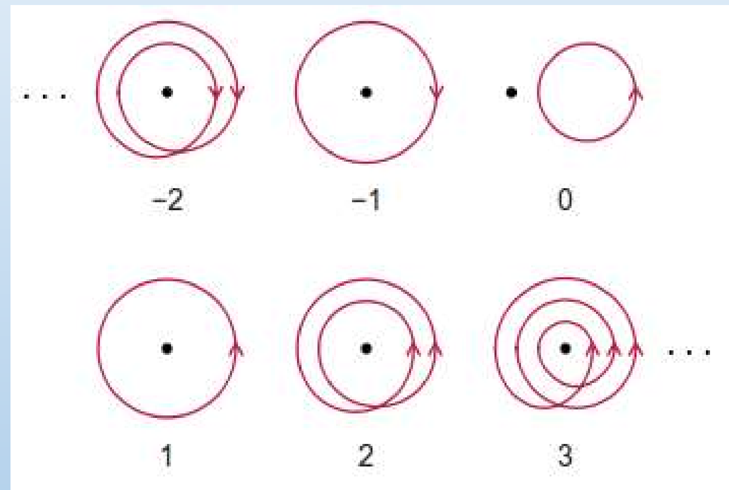
- Bulk-Boundary Relationship plays a Central role in Topology
  (Homology)
- Topological Inumbrer/nvariant can be expressed as an Integral
  (Differential Topology: Cohomology and Homotopy)

⟹ Integral Calculus is Important in Curved Manifold without Metric

Winding Numbers I ($S^1 \to S^1$)

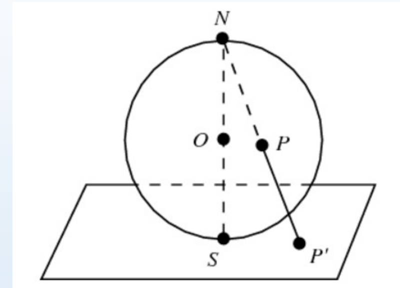
Contour integral in
complex analysis

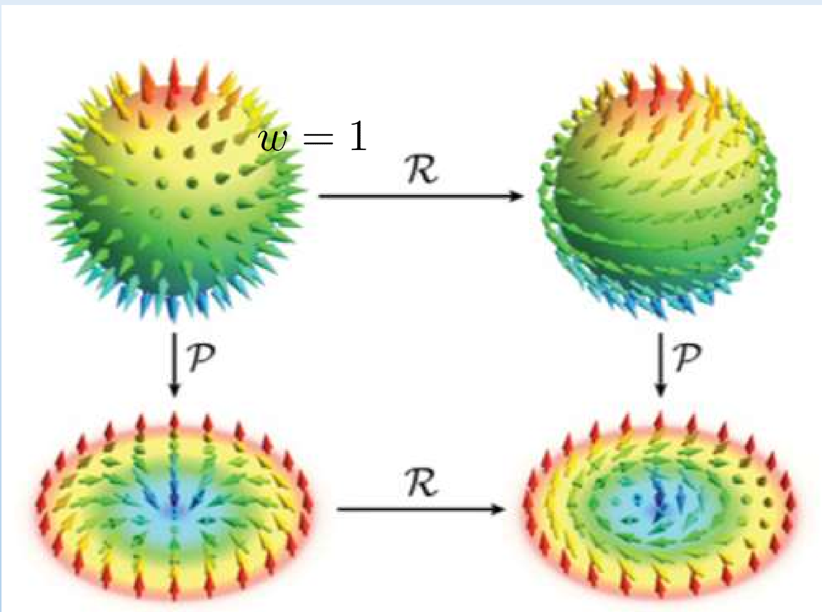




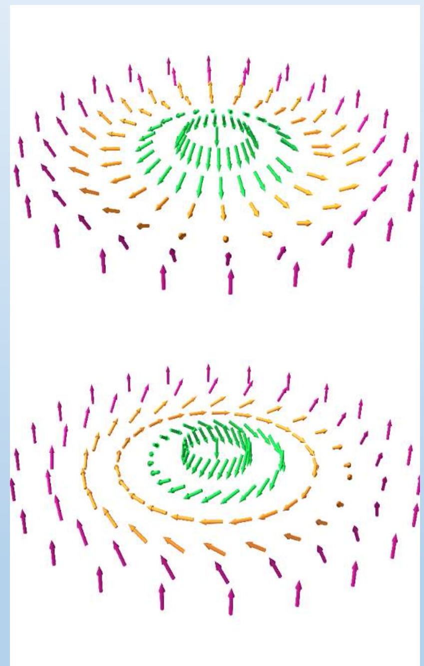
$$w = \frac{1}{2\pi}\int_0^{2\pi} d\theta \, \frac{d\theta'}{d\theta} = integer$$

Winding Number II ($S^2 \to S^2$)

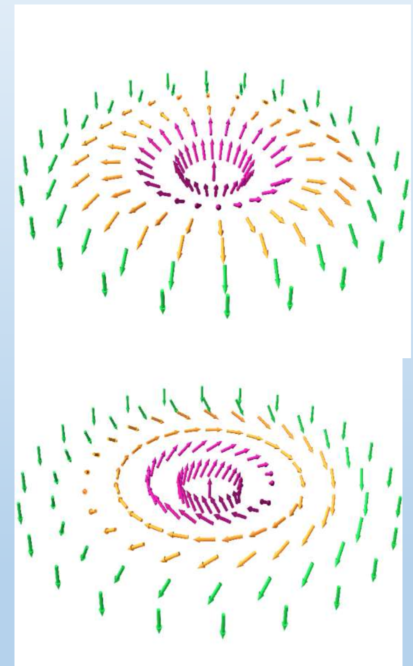




$$w = \frac{1}{4\pi} \int dxdy\, \boldsymbol{n} \cdot \left(\frac{\partial \boldsymbol{n}}{\partial x} \times \frac{\partial \boldsymbol{n}}{\partial y} \right)$$



$w = 1$



$w = -1$

A Primer of Topology for Physicists

- Manifold and Differential Forms
  Stokes Theorem
- Mapping Degree/Winding Numbers
  Kronecker Invariant, WZW Action
- Topology in Gauge Theory
  Aharonov-Bohm Effect and Flux Quantum
  Dirac Quantization for Magnetic Monopole
  Instantons and Theta Vacuum
  Index Theorem for Zero Modes and Anomaly
- Descent Equations:
  Relations in Different Dimensions

Topology

Definition:Properties that remain unchanged under continuous (or smooth) deformation.

1. Combinatoric Topology: Pure Math
2. Algebraic Topology:
   (1)Homotopy group, (2)Homology group, (3)Cohomology group
3. Differential Topology:
   Differential forms $\Rightarrow$ Integral topological invariants $\Rightarrow$ Topological number

In physics we need to get a number to be compared with experiments, so topological numbers obtained by integral invariants are very useful in physics, though not every topological number can be expressed as an integral.

Integration over a smooth manifold

Manifolds are generalization of curves, surfaces, hyper-surfaces, etc.
A manifold (with dimension n) is defined by the following data:
1. Local coordinates patches (with n coordinates in each patch)

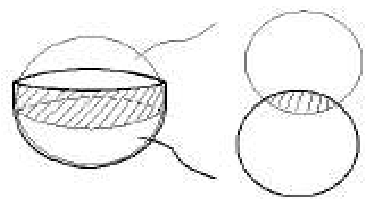


Fig: local coordinates for a 2-sphere

2. In overlapping region(s), the coordinate transformations

$$x'^{\mu} = f^{\mu}(x^1, x^2, \cdots, x^n), \qquad (\mu = 1, 2, \cdots, n)$$

must be smooth.
3. Collection of all admissible coordinates patches that covers the whole manifold, a differential structure).
Here, properties 2 and 3 contain global information.

Integration elements

In $\mathbb{R}^3$, we have the following integrations.
(1) Line integral $\int_c \vec{A} \cdot d\vec{x}$
(2) Surface integral $\int_S \vec{E} \cdot d\vec{\sigma}$
(3) Volume integral $\int_V f(\vec{x}) d\tau$
In higher dimensions, the surface and volume elements are not vector and scalar but anti-symmetric tensors.

$$
\begin{aligned}
d\vec{\sigma} \quad &\rightarrow \quad d\sigma^{\mu\nu} = dx^\mu \wedge dx^\nu \\
d\tau \quad &\rightarrow \quad d\tau^{\mu\nu\lambda} = dx^\mu \wedge dx^\nu \wedge dx^\lambda
\end{aligned}
$$

Here, for example, an volume element formed by three infinitesimal non-planar vectors δu_a $(a = 1, 2, 3)$ is understood as

$$
dx^\mu \wedge dx^\nu \wedge dx^\lambda \rightarrow \epsilon^{abc} (\delta u_a)^\mu (\delta u_b)^\nu (\delta u_c)^\lambda \ . \tag{1}
$$

Differential forms as "integrands"

The differential k-form (of degree k) is

$$\omega = \frac{1}{k!}\omega_{\mu_1\ldots\mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k} \;;$$

where $\omega_{\mu_1\ldots\mu_k}$ and $dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$ are totally anti-symmetric, respectively. We have following properties:

(1) $dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu$. More generally, $\omega_1 \wedge \omega_2 = (-1)^{deg(\omega_1)\cdot deg(\omega_2)}\omega_2 \wedge \omega_1$.

(2) Define: $d\omega = \frac{1}{k!}\frac{\partial \omega_{\mu_1\ldots\mu_k}}{\partial x^\mu} dx^\mu dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$.
Namely we have $d \equiv \frac{\partial}{\partial x^\mu} dx^\mu \wedge$.

(3) $d^2 = 0$ (Most important property)

(4) Leibniz Rule: $d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 + (-1)^{deg\omega_1}\omega_1 \wedge (d\omega_2)$

Differential Forms (cont.)

(5) Given a map $f : M \to N, (x^\mu \mapsto y^\mu)$, then for a k-form ω on N we obtain a form

$$
\begin{aligned}
f^*\omega &= \frac{1}{k!}\omega_{\alpha_1,\dots\alpha_k}(y(x))dy^{\alpha_1}(x)\wedge\dots\wedge y^{\alpha_k}(x) \\
&= \frac{1}{k!}\omega_{\alpha_1,\dots\alpha_k}(y(x))\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\dots\frac{\partial y^{\alpha_k}}{\partial x^{\mu_k}}dx^{\mu_1}\wedge\dots\wedge dx^{\mu_k}
\end{aligned}
$$

defined on M. This operation is called **Pull-Back**.

(6) Stoke's Theorem describes the Bulk-Boundary Relation:

$$\int_V d\omega = \int_{\partial V}\omega$$

Corollary: If ω is a closed form, i.e. $d\omega = 0$, then $\int_S \omega$ is unchanged under smooth deformation of the compact manifold S. (A compact manifold has no boundary.)

Therefore, a closed form is always associated with an integral topological invariant. Useful topological integral invariants include mapping degrees, or winding numbers, and Chern numbers.

Stokes' Theorem in 3 dimensions

For 0-form $f(\vec{x}) \equiv f(x,y,z)$, then 1-form $df = \frac{\partial f}{\partial x^i} dx^i$

For 1-form $A = A_i(\vec{x}) dx^i$, we have two possible derivations

Curl (2-form) : $dA = \frac{1}{2} \frac{\partial A_j}{\partial x^i} dx^i \wedge dx^j$, $(dA)_{ij} = \epsilon_{ijk} (\nabla \times \vec{A})_k$

Dual (2-form): $*A = \frac{1}{2} \epsilon_{ijk} A_k dx^i \wedge dx^j,$

Divergence (0-form):

$$d * A = \frac{\partial}{\partial x^l} A_k (\frac{1}{2} \epsilon_{ijk}) dx^l \wedge dx^i \wedge dx^j$$

$$*(d * A) = \epsilon_{ijl} (d * A)_{ijl} = \nabla \cdot \vec{A}$$

Stokes Theorem unifies the usual Green's, Stokes's and Gausss's Theorems:

Mapping Degree

$$\varphi : M \to N \quad (dim(M) = dim(N),\ M, N \text{ are compact and oriented})$$

Intuition: The image of $\varphi(M)$ must cover N an integer times.
Consider $\varphi : S^1 \to S^1$

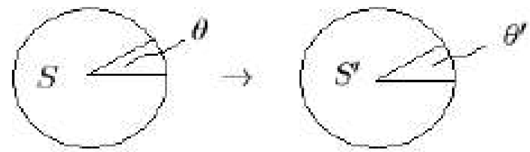


Fig: Map $S^1 \to S^1$

$$\begin{aligned}
&(1).\theta \mapsto \theta' = \theta && (0 \le \theta < 2\pi), \text{ it covers 1 time.} \\
&(2).\theta \mapsto \theta' = 2\theta && (0 \le \theta < 2\pi), \text{ it covers 2 time.} \\
&(3).\theta \mapsto \theta' = \theta && (0 \le \theta < \pi\ ) \\
&\qquad\qquad\quad = 2\pi - \theta && (\pi \le \theta < 2\pi)
\end{aligned}$$

it covers 0 times.

The (integer) number of times that $f(M)$ covers N can be expressed by an integral mathematically.

Mapping degree (cont.)

Mapping degree or winding number:

$$deg(\varphi) = \frac{\int_M \varphi^* \omega}{\int_N \omega}$$

where ω is n-form on N (Volume form).
Example 1. **Kronecker Integral**: $N = S^n$
$\varphi : M \to S^n$ (S^n is unit sphere $(\sum_{\alpha=1}^{n+1} (y^\alpha)^2 = 1)$.)
So we have $\sum_\alpha^{n+1} (\varphi^\alpha(x))^2 = 1$ (unit vector in $\mathbb{R}^{n+1}$)
Take $\omega =$ to be the volume element

$$deg(\varphi) = \frac{1}{V_n} \int d^n x \frac{1}{n!} \varepsilon^{\mu_1 \dots \mu_n} \varepsilon_{\alpha_1 \dots \alpha_{n+1}} \varphi^{\alpha_1}(x) \frac{\partial \varphi^{\alpha_2}(x)}{\partial x^{\mu_1}} \dots \frac{\partial \varphi^{\alpha_{n+1}}(x)}{\partial x^{\mu_n}}$$

with $V_n = \frac{(n+1)\pi^{\frac{n+1}{2}}}{\Gamma[\frac{1}{2}(n+3)]}$.

Properties of Mapping Degree

We have the following properties:
Homotopy Invariant: $deg(\varphi)$ is homotopic invariant.
Definition: We say $\varphi_1 \sim \varphi_2$ if exists smooth $F: M \times I \to N$, such that

$$F(x, t = 0) = \varphi_1(x), \quad F(x, t = 1) = \varphi_2(x)$$

(F represents the process in which φ_1 is deformed to φ_2)
$\varphi_1 \sim \varphi_2 \;\Rightarrow\; deg(\varphi_1) = deg(\varphi_2)$
Note. The converse is NOT true generally, but true in Kronecker's case. We have **Hopf Theorem.** Suppose M is connected, compacted and oriented. For maps $\varphi_1, \varphi_2: M \to S^n, (dim(M) = n)$. We have

$$\varphi_1 \sim \varphi_2 \Leftrightarrow deg(\varphi_1) = deg(\varphi_2)$$

Homotopy Integral Invariant (cont.)

Example 2: The case when $N = G$ (Lie Group)
Here $\dim(M) \neq \dim(N)$ is allowed.
Consider a map $g : M \to G$, $(x \mapsto g(x), g(x) \in G)$. The form

$$Tr\{[g^{-1}(x)dg(x)]^k\}$$

is the pull-back on M of $[g^{-1}dg]^k$ on G. Here $g^{-1}dg$ is the Cartan-Maurer 1-form on G:

$$g^{-1}dg = \sum_{a=1}^{r} T^a V_\alpha^a(g) dg^\alpha$$

Where $r = dim(G)$ and g^α are coordinates on G. So the pull-back is given by

$$g^{-1}(x)dg(x) = \sum_{a=1}^{r} T^\alpha \widetilde{V}_\mu^a(g(x)) dx^\mu \ ,$$

where $\widetilde{V}_\mu^a = V_\alpha^a(g(x))\frac{\partial g^\alpha}{\partial x^\mu}$.

Homotopy Integral Invariant (cont.)

Let us define
$w[g(x)] = c_k \int_M Tr\{[g^{-1}(x)dg(x)]^k\}$.
Lemma. $Tr(g^{-1}dg)^k = 0, \quad$ *if* $k = even$;
$\mathrm{dTr}(\mathrm{g}^{-1}dg)^k = 0, \quad$ *if* $k = odd$.
When M is a k-sphere, S^k with $k = odd$, $c_k = \frac{(\frac{k-1}{2})!}{(2\pi)^{\frac{k+1}{2}} k!}$.
Theorem. $w[g(x)] = integer$.
Application: In gauge theories, a map $g : M \to G$ is called a gauge transformation and $w(g)$, if non-zero, is called the winding number of the *large gauge transformation* f.
Example 3: $M = S^3, N = SU(2)$. *In this case,* $w(g)$ *can be calculated by either of the above formulas.*
Let us define $(g_1 \circ g_2)(x) = g_1(x) \cdot g_2(x)$. *Then we have the following property:*

$$w(g_1 \circ g_2) = w(g_1) + w(g_2).$$

$Tr[(g_1 g_2)^{-1} d(g_1 g_2)]^3 = Tr[(g_1^{-1} dg_1)]^3 + Tr[(g_2^{-1} dg_2)^3] + d(\text{something})$.

Reference Books

1. H. Flanders, “Differential Forms with Applications to the Physical Sciences” , (Dover Publications, 1989)
2. C. Nash and S. Sen, “Topology and Geometry for Physicists”, (Academic Press, Inc. , London, 1983)

These are suitable for this class. For advanced textbooks see, e.g.,

3. Y. Choquet-Bruhat, C. Dewitt-Morette and M. Dillard-Bleick, “Anaysis, Manifolds and Physics” , Revised Edition, (North-Holland, Amsterdam, 1982)

4. T. Frankel, “The Geometry of Physics -- An Introduction”, (Cambridge University Press, 1997)

Problems

1. Given $\alpha = 2yzdx + x^2dy + xyzdz$, $\beta = \sin x dx + \cos y dy$, compute (1) $\alpha \wedge \beta$, and (2) $d\alpha$.

2. Given α and β as given above, explicitly compute (3) $d(d\alpha)$ and (4) $d(\alpha\wedge\beta)$.

3. If further $\gamma = 3z^3dx+(x^2 + y^2 + z^2)dy+5y^2\, dz$, compute $\alpha\wedge\beta\wedge\gamma$ =?

4. Consider a 2-sphere S^2, described by $x^2 + y^2 + z^2 = 1$.
   Show that its surface element is given by $\beta = x dy\wedge dz + y dz\wedge dx + z dx\wedge dy$.
   Hint: Use the polar coordinates: $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$.
   prove that $\beta = \sin\theta d\theta\wedge d\phi$, $d\beta = 3dx\wedge dy\wedge dz$.

End